Y13 – Chapter 1 Algebraic Methods

What do I need to be able to do?

By the end of this chapter you should be able to:

- Use proof by contradiction to prove true statements
- Add, subtract, multiply and divide two or more algebraic fractions
- Convert an expression with linear factors in the denominator into partial fractions
- Convert an expression with repeated linear fractions in the denominator into partial fractions
- Divide algebraic expressions
- Convert an improper fraction into partial fraction form

Partial Fractions

Sometimes it can be useful to split a single algebraic fraction into two or more partial fractions.

Eg: $\frac{7x-13}{(x-3)(x+1)} = \frac{2}{x-3} + \frac{5}{x+1}$

When solving partial fractions, you start by setting your function equal to the unknown fractions you are trying to find. There are 3 different layouts which depend on the starting function

1) All linear terms in the denominator

$$\frac{7x-13}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

2) A repeated term in the denominator:

$$\frac{3x^2+7x-12}{(x-5)(x+2)^2} = \frac{A}{(x-5)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

3) *Improper fractions:

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = A + \frac{B}{(x-1)} + \frac{C}{(x-2)}$$

Steps to solve:

1) Sett your functions equal to the correct unknown fraction as above
2) Add the fractions using a common denominator (this should be the same as the original denominator)
3) Set the numerators as equal
4) Substitute values for x that will, in turn, make each bracket zero and/or equate coefficients to create enough equations to find the values of A, B, C etc

*NB: You can either use algebraic division or the relationship F(x)= Q(x) x divisor + remainder to convert an improper fraction into a mixed fraction

Key words:

- Contradiction – a disagreement between two statements which means that both cannot be true.
- Coefficient – A number used to multiply by a variable
- Improper algebraic fraction – One whose numerator has a degree equal to or larger than the denominator. It must be converted to a mixed fraction before you can express it in partial fractions

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Proof by Contradiction

To prove by contradiction you start by assuming that the statement is false. You then use logical steps until you contradict yourself by leading to something that is impossible. You can then conclude that your assumption was incorrect and that the original statement was true.

Eg: Prove by contradiction that $\sqrt{2}$ is irrational

Assumption: $\sqrt{2}$ is rational, therefore $\sqrt{2}$ can be written as $\frac{a}{b}$ where a and b are in their lowest form and that $\frac{a}{b}$ is in its lowest terms

$$\therefore 2 = \left(\frac{a}{b}\right)^2$$

$$2 = \frac{a^2}{b^2}$$

$$\therefore 2b^2 = a^2$$

This means that a^2 is even which means that a is even.
If a is even then it can be expressed as 2k

$$\therefore a^2 = 2b^2$$

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2$$

$$2k^2 = b^2$$

This means that b^2 is even which means that b is even.

Conclusion: If a and b are both even then the have a common factor of 2 so $\frac{a}{b}$ cannot be a fraction in its lowest terms which is a contradiction. This means that the original assumption is not correct and therefore $\sqrt{2}$ is irrational

Y13 – Chapter 2 Functions and Graphs

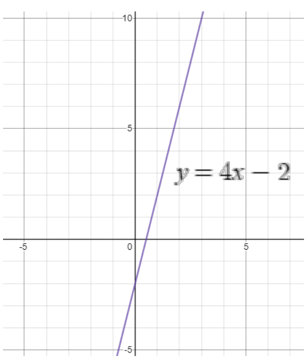
What do I need to be able to do?

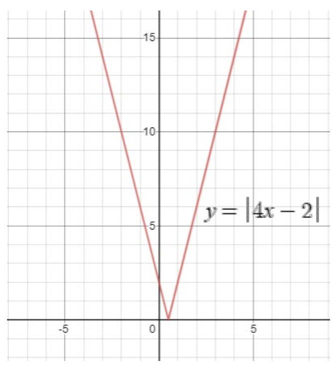
By the end of this chapter you should be able to:

- Understand and use the modulus function
- Understand mappings and functions, and use domain and range
- Combine two or more functions to get a composite function
- Know how to find the inverse of a function both graphically and algebraically
- Sketch the graphs of the modulus function
- Apply a combination of transformations to a curve
- Transform a modulus function

The Modulus Function

To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ and then reflect any section of the graph that is below the x-axis in the x-axis





When solving modulus equations algebraically you consider the positive and negative argument (the function inside the modulus) separately

Eg:

Solve $|2x - 1| = 5$

$2x - 1 = 5$

$2x = 6$

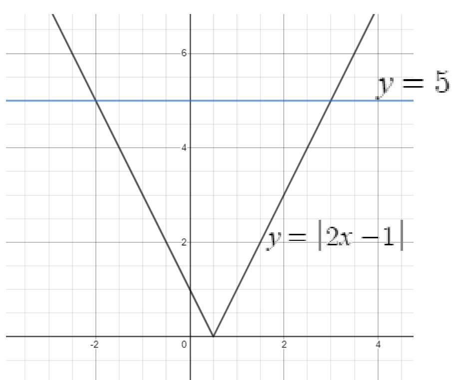
$x = 3$

$-(2x - 1) = 5$

$-2x + 1 = 5$

$-2x = 4$

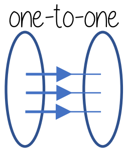
$x = -2$



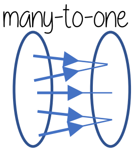
Functions and Mappings

A mapping is a function if each input has a distinct output. Functions can either be one-to-one or many-to-one

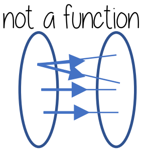
one-to-one



many-to-one



not a function



Key words:

- Modulus – the absolute value or modulus of a real number x, denoted $|x|$, is the non-negative value of x without regard to its sign. For example, the absolute value of 3 is 3, and the absolute value of –3 is also 3.
- Composite function – A function made of other functions, where the output of one is the input to the other
- Inverse function – An inverse function is a function that undoes the action of another function

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Composite Functions

Always apply the inside function first.

To find fg(x) do g(x) first then substitute your answer into f(x) to find the answer

Eg: $f(x) = x^2$ and $g(x) = x + 1$

a) Find fg(2)

$g(2) = 2 + 1 = 3$

$f(3) = 3^2 = 9$

b) Find gf(x)

$f(x) = x^2$

$g(x^2) = x^2 + 1$

The Inverse Function

The inverse of a function performs the opposite operation to the original function. Inverse functions only exist for one-to-one functions.

The inverse of a function f(x) is written as $f^{-1}(x)$

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$

The domain of f(x) is the range of $f^{-1}(x)$

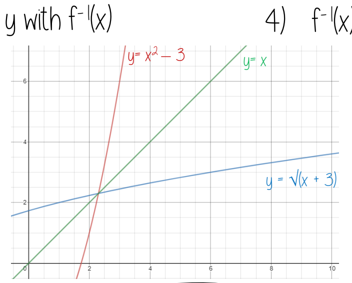
The range of f(x) is the domain of $f^{-1}(x)$

To find the inverse function:

1) Write it as y =
2) Swap x and y
3) Rearrange to make y the subject
4) Replace y with $f^{-1}(x)$

$f(x) = x^2 - 3$ find $f^{-1}(x)$

1) $y = x^2 - 3$
2) $x = y^2 - 3$
3) $\sqrt{(x + 3)} = y$
4) $f^{-1}(x) = \sqrt{(x + 3)}$



Y13 – Chapter 3 Sequences and Series

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What do I need to be able to do?

By the end of this chapter you should be able to:

- Find the nth term of an arithmetic sequence and a geometric sequence
- Prove and use the formula for the sum of the first n terms of an arithmetic series
- Prove and use the formula for the sum of a finite geometric series
- Prove and use the formula for the sum to infinity of a convergent geometric series
- Use sigma notation
- Generate sequences from recurrence relations
- Model real life situations

Key words:

- Sequence – A list of numbers or objects in a special order
- Series – The sum of terms in a sequence
- Arithmetic sequence – A sequence made by adding the same value each time
- Geometric sequence – A sequence made by multiplying by the same value each time. There is a common ratio between consecutive terms
- Arithmetic series– the sum of the terms of an arithmetic sequence
- Geometric series – the sum of the terms in a geometric sequence
- Common ratio – The amount we multiply by each time in a geometric sequence
- Converging sequence/series – A sequence/series converges when it keeps getting closer and closer to a certain value
- Divergent series – does not settle towards a certain value. When a series diverges it goes off to infinity, minus infinity, or up and down without settling towards some value.

Arithmetic Sequences and Series

The formula for the nth term of an arithmetic sequence is:

$$u_n = a + (n-1)d$$

u_n is the nth term
a is the first term
d is the common difference

The formula for the sum of the first n terms of an arithmetic series is:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

It can also be written as:

$$S_n = \frac{n}{2}(a + l)$$

a is the first term
d is the common difference
l is the last term

Geometric Sequences and Series

The formula for the nth term of a geometric sequence is:

$$u_n = ar^{n-1}$$

u_n is the nth term
a is the first term
r is the common ratio

The formula for the sum of the first n terms of a geometric series is:

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$$

It can also be written as:

$$S_n = \frac{a(r^n-1)}{r-1}, r \neq 1$$

a is the first term
r is the common ratio

Sigma Notation

Σ means "the sum of". You write on the top and bottom to show which terms you are summing.

Eg:

$$\sum_{r=1}^{5}(2r-3) = -1 + 1 + 3 + 5 + 7$$

This tells you that you are summing the expression in brackets with r=1 up to r=5

Substitute r=1, r=2, r=3, r=4 and r=5 into the expression in brackets to find the 5 terms in this arithmetic series

Sum to Infinity

As n tends to infinity, the sum of a geometric series is called the sum to infinity.
A geometric series is convergent only when $|r| < 1$, where r is the common ratio

The formula for the sum to infinity of a convergent series is:

$$S_\infty = \frac{a}{1-r}$$

a is the first term
r is the common ratio

Recurrence Relations

The next term in the sequence is the function of the previous term

$$u_{n+1} = f(u_n)$$

What do I need to be able to do?

By the end of this chapter you should be able to:

- Expand $(1+x)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid
- Expand $(a+bx)^n$ for any rational constant n and determine the range of values of x for which the expansion is valid
- Use partial fractions to expand fractional expressions

Y13 – Chapter 4 Binomial Expansion

Key words:

- Infinite series – The sum of infinite terms that follow a rule

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The Binomial Expansion

$(a + b)^n$ when n is a fraction or a negative number (ie. NOT a positive integer) there will be an infinite number of terms. This means that the binomial expansion can only be used when $-1 < x < 1$.

When n is a fraction or a negative number the following form of the binomial expansion should be used:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{r}x^r + \cdots, (|x| < 1, n \in \mathbb{R})$$

The expansion is valid when $|x| < 1$

The expansion of $(1 + bx)^n$ is valid for $|bx| < 1$ or $|x| < \frac{1}{b}$

We can use the expansion of $(1 + x)^n$ to expand $(a + bx)^n$ by taking out a factor of a^n out of the expression

$$(a+bx)^n = (a\left(1 + \frac{b}{a}x\right))^n = a^n(1 + \frac{b}{a}x)^n$$

The expansion of $(a+bx)^n$ is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \frac{a}{b}$

Partial Fractions

We can use partial fractions to simplify the expansions of more difficult expressions

E.g.

a) Express $\frac{4-5x}{(1+x)(2-x)}$ as partial fractions

b) Hence show that the cubic approximation of $\frac{4-5x}{(1+x)(2-x)}$ is $2 - \frac{7x}{2} + \frac{11}{4}x^2 - \frac{25}{8}x^3$

c) State the range of values of x for which the expansion is valid

a) $\frac{4-5x}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$

$\equiv \frac{A(2-x)+B(1+x)}{(1+x)(2-x)}$

$4 - 5x \equiv A(2-x) + B(1+x)$

Substitute x = 2:

$4 - 10 = A \times 0 + B \times 3$

$B = -2$

Substitute x = -1:

$4 + 5 = A \times 3 + B \times 0$

$A = 3$

b) $\frac{4-5x}{(1+x)(2-x)} = \frac{3}{1+x} - \frac{2}{2-x}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

The expansion of $3(1+x)^{-1} = 3 - 3x + 3x^2 - 3x^3 + \cdots$

The expansion of $2(2-x)^{-1} = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots$

Hence $\frac{4-5x}{(1+x)(2-x)} = (3 - 3x + 3x^2 - 3x^3) - \left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}\right)$

$= 2 - \frac{7}{2}x + \frac{11}{4}x^2 - \frac{25}{8}x^3$

c) $\frac{3}{1+x}$ is valid if $|x| < 1$ $\quad$ $\frac{2}{2-x}$ is valid if $|x| < 2$

So the expansion is valid when $|x| < 1$

What do I need to be able to do?

By the end of this chapter you should be able to:

- Convert between degrees and radians
- Know exact values of angles measured in radians
- Find arc length using radians
- Find areas of sectors and segments using radians
- Solve trigonometric equations
- Use small angle approximations

Y13 – Chapter 5 Radians

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Key words:

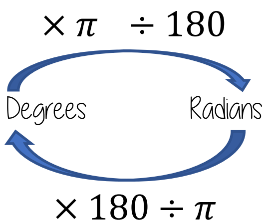
- Radian – The angle made by taking the radius and wrapping it round the circle
- Arc length – The distance along part of the circumference of a circle, or of any curve
- Sector – the area between two radiuses and the connecting arc of a circle
- Segment – The smallest part of a circle made when it is cut by a line

Converting between degrees and radians

2π radians = 360°

π radians = 180°

1 radian = $180/\pi$



Arc lengths, Sectors and Segments

When working in radians:

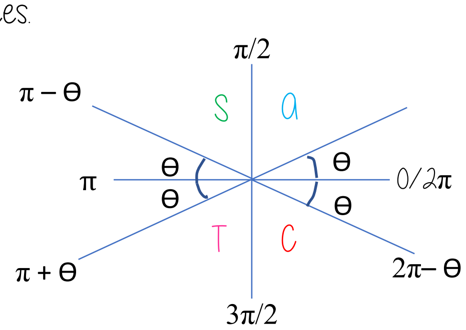
Arc length = $r\theta$

Area of sector = $\frac{1}{2}r^2\theta$

Area of a segment = $\frac{1}{2}r^2(\theta - \sin\theta)$

Solving Trigonometric Equations

This works the same way as solving trigonometric equations in degrees.

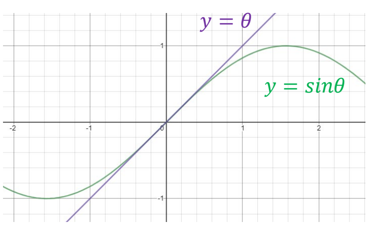


$$\sin\theta = \sin(\pi - \theta)$$
$$\cos\theta = \cos(2\pi - \theta)$$
$$\tan\theta = \tan(\pi + \theta)$$
$$-\sin\theta = \sin(\pi + \theta) = \sin(2\pi - \theta)$$
$$-\cos\theta = \cos(\pi - \theta) = \cos(\pi + \theta)$$
$$-\tan\theta = \tan(\pi - \theta) = \tan(2\pi - \theta)$$

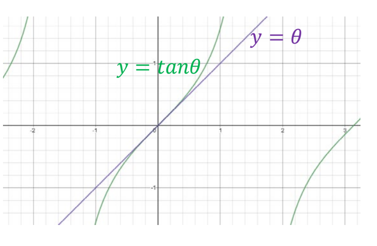
Small Angle Approximations

When θ is small and measured in radians:

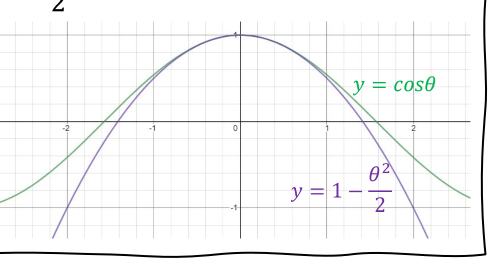
$\sin\theta \approx \theta$



$\tan\theta \approx \theta$



$\cos\theta \approx 1 - \frac{\theta^2}{2}$



Y13 – Chapter 6 Trigonometric Functions

What do I need to be able to do?

By the end of this chapter you should be able to:

- Understand secant, cosecant and cotangent and their relationship to cosine, sine and tangent
- Understand the graphs of secant, cosecant and cotangent
- Simplify expressions, prove identities and solve equations involving secant, cosecant and cotangent
- Understand and use inverse trigonometric functions

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Key words:

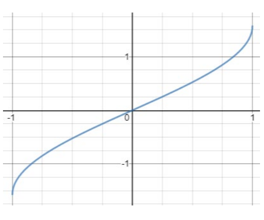
- Cosecant – In a right angled triangle, the cosecant of an angle is: The length of the hypotenuse divided by the length of the side opposite the angle
- Secant– In a right angled triangle, the secant of an angle is: The length of the hypotenuse divided by the length of the adjacent side.
- Cotangent – In a right angled triangle, the cotangent of an angle is: The length of the adjacent side divided by the length of the side opposite the angle

Trig Identities

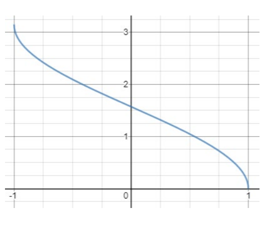
$$1 + tan^2x \equiv sec^2x$$
$$1 + cot^2x \equiv cosec^2x$$

Graphs of Inverse Functions

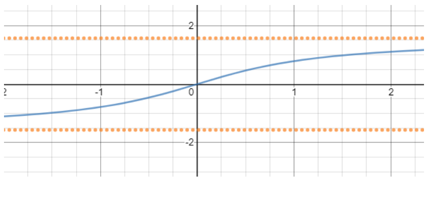
The inverse of sin(x) is arcsin(x)
The domain is $-1 \leq x \leq 1$
The range is $-\pi/2 \leq \arcsin(x) \leq \pi/2$



The inverse of cos(x) is arccos(x)
The domain is $-1 \leq x \leq 1$
The range is $0 \leq \arcsin(x) \leq \pi$



The inverse of tan(x) is arctan(x)
The domain is $x \in \mathbb{R}$
The range is $-\pi/2 \leq \arctan(x) \leq \pi/2$

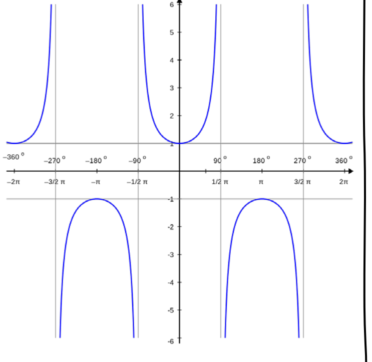


Secant, Cosecant and Cotangent

$$\sec x = \frac{1}{\cos x}$$

The graph has symmetry in the y-axis, a period of 360/2 π
It has vertical asymptotes at all of the values for which cos(x)=0

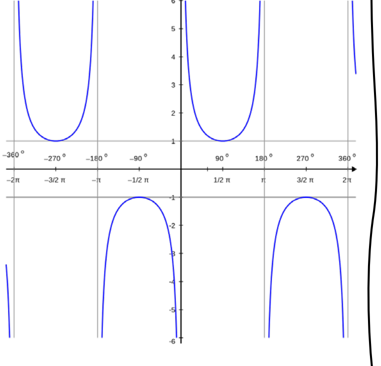
The domain of y=sec(x) is $x \in \mathbb{R}$
In degrees:
$x \neq 90, 270, 450$... (any odd multiple of 90)
In radians:
$x \neq \pi/2, 3\pi/2, 5\pi/2$... (any odd multiple of $\pi/2$)
The range of y=sec(x) is $y \leq -1$ or $y \geq 1$



$$cosec\ x = \frac{1}{\sin x}$$

The graph has symmetry in the y-axis, a period of 360/2 π
It has vertical asymptotes at all of the values for which cos(x)=0

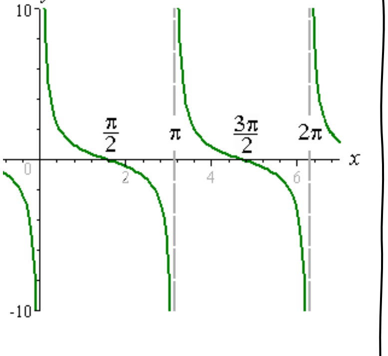
The domain of y=sec(x) is $x \in \mathbb{R}$
In degrees:
$x \neq 90, 270, 450$... (any odd multiple of 90)
In radians:
$x \neq \pi/2, 3\pi/2, 5\pi/2$... (any odd multiple of $\pi/2$)
The range of y=sec(x) is $y \leq -1$ or $y \geq 1$



$$cot\ x = \frac{1}{tan\,x}$$

The graph has vertical asymptotes at all of the values for which tan(x)=0

The domain of y=cot(x) is $x \in \mathbb{R}$
In degrees:
$x \neq 0, 180, 360$... (any odd multiple of 180)
In radians:
$x \neq 0, \pi, 2\pi$... (any multiple of π)
The range of y=sec(x) is $y \in \mathbb{R}$

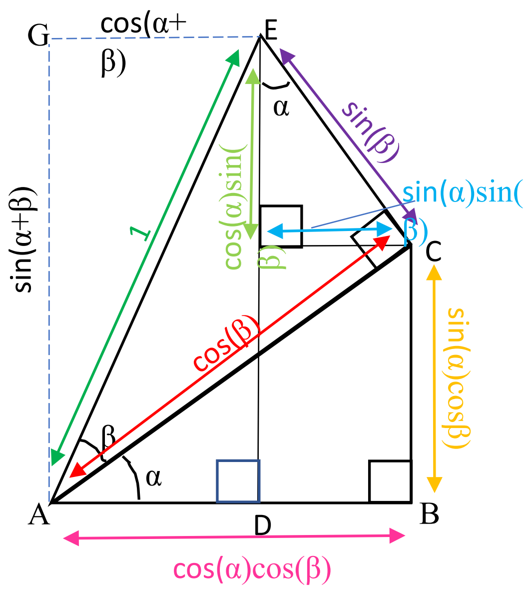


What do I need to be able to do?

By the end of this chapter you should be able to:

- Prove and use the addition formulae
- Understand and use the double angle formulae
- Solve trigonometric equations using addition and double angle formulae
- Write expressions in the form Rcos(θ±α) and Rsin(θ±α)
- Prove trigonometric identities
- Model real life situations

Proof of the Addition Formulae



Using the properties of sine and cosine we can label the diagram as above.

Using triangle ADE:
DE= sin(α+β)
AD = cos (α+β)

DE = DF + FE
∴ sin(α+β) = sin(α)cos(β) + cos(α)sin(β)

AD = AB – DB
∴ cos(α+β) = cos(α)cos(β) – sin(α)sin(β)

Y13 – Chapter 7 Trigonometry and Modelling

Addition Formulae

Sometimes are known as the compound angle formulae

$$sin(A+B) \equiv sinAcosB + cosAsinB$$
$$sin(A-B) \equiv sinAcosB - cosAsinB$$

$$cos(A+B) \equiv cosAcosB - sinAsinB$$
$$cos(A-B) \equiv cosAcosB + sinAsinB$$

$$tan(A+B) \equiv \frac{tanA + tanB}{1 - tanAtanB}$$

$$tan(A-B) \equiv \frac{tanA - tanB}{1 + tanAtanB}$$

Simplifying acos(x)± bcos(x)

Sometimes known as the harmonic form. You can write expressions in the form acos(x) ± bcos(x) as a function of sine or cosine only.

$acos\theta \pm bsin\theta$ can be written as either:

$Rsin(x \pm \alpha)$ where R>0 and 0< α<90

$Rcos(x \pm \beta)$ where R>0 and 0< β<90

Where $Rcos\alpha = a$ and $Rsin\alpha = b$ and $R = \sqrt{a^2 + b^2}$

Double Angle Formulae

You can use the addition formulae to derive the following double angle formulae:

$$sin(2A) \equiv 2sinAcosA$$

$$cos(2A) \equiv cos^2A - sin^2A \equiv 2cos^2A - 1 \equiv 1 - 2sin^2A$$

$$tan(2A) \equiv \frac{2tanA}{1 - tan^2A}$$

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Y13 – Chapter 8 Parametric Equations

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Key words:

- Cartesian equations – Gives a direct relationship between x and y
- Parametric equations – Uses a third variable (usually t or θ) to define x and y

What do I need to be able to do?

By the end of this chapter you should be able to:

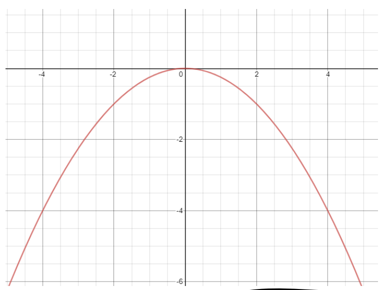
- Convert parametric equations into Cartesian form
- Understand and use parametric equations of curves and sketch parametric curves
- Solve problems involving parametric equations
- Use parametric equations in modelling

Sketching parametric equations of curves

When sketching a parametric equation, sub in values of t to find x and y values and then sketch as normal!

Sketch the curve defined by $x=2t$ and $y=-t^2$ between $t=-3$ and 3.

T	-3	-2	-1	0	1	2	3
X	-6	-4	-2	0	2	4	6
Y	-9	-4	-1	0	-1	-4	-9



Calculus with parametric equations*

* This section actually appears in your text book in chapters 9 and 11

Differentiation:

If x = f(t) and y = g(t)

Then:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Integration:

If x = f(t) and y = g(t)

Then:

$$\int y\,dx = \int y\frac{dx}{dt}\,dt$$

Remember to adjust limits if you are using definite integration

Converting between parametric and cartesian equations

Combine the two equations by rearranging one of them to make t the subject and then substitute into the other equation.

OR

Rearrange both equations to make t the subject and then equate the two equations

Eg: Convert the following parametric equations into cartesian form

$x = t + 3 \quad y = 2t^2$

$x = t + 3 \rightarrow t = x - 3$

$y = 2(x - 3)^2$

OR

$x = t + 3 \quad y = 2t^2$

$x = t + 3 \rightarrow t = x - 3$

$y = 2t^2 \rightarrow t = \sqrt{(y/2)}$

$\sqrt{(y/2)} = x - 3$

$y/2 = (x - 3)^2$

$y = 2(x - 3)^2$

If your parametric equations contains trigonometric functions, first find an identity that connects then rearrange the parametric equations so that you can substitute into the identity

Domain and range

For parametric equations x = p(t) and y = q(t) with Cartesian equation y = f(x)

- The domain of f(x) is the range of p(t)
- The range of f(x) is the range of q(t)

Y13 – Chapter 9 Differentiation

Pure Maths Year 2

Key words:

- Concave – Curves inwards
- Convex – Curves outwards

What do I need to be able to do?

By the end of this chapter you should be able to:

- Differentiate trigonometric functions
- Differentiate exponentials and logarithms
- Use the chain, product and quotient rules
- Use the second derivative to describe a function's behaviour
- Solve problems involving connected rates of change
- Construct differential equations
- Differentiate parametric equations (see parametric equations sheet)

Differentiating Trigonometric functions

If $y = \sin kx$, then $\frac{dy}{dx} = k\cos kx$

If $y = \cos kx$, then $\frac{dy}{dx} = -k\sin kx$

If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$

If $y = \operatorname{cosec} kx$, then $\frac{dy}{dx} = -k\operatorname{cosec} kx\cot kx$

If $y = \sec kx$, then $\frac{dy}{dx} = -k\sec kx\tan kx$

If $y = \cot kx$, then $\frac{dy}{dx} = -k\operatorname{cosec}^2 kx$

If $y = \arcsin x$, then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

If $y = \arccos x$, then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

Implicit Differentiation

Use when equations are difficult to rearrange into the form y = f(x)

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

Second Derivatives

The function f(x) is concave on a given interval if and only if f''(x) ≤ 0 for every value of x in the value in that interval

The function f(x) is convex on a given interval if and only if f''(x) ≥ 0 for every value of x in that interval

A point of inflection is a point at which f''(x) changes sign

Chain, Product and Quotient Rules

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Where y is a function of u and u is another function of x

In function notation:

If y = (f(x))^n then $\frac{dy}{dx} = n(f(x))^{n-1}f'(x)$

If y = f(g(x)) then $\frac{dy}{dx} = f'(g(x))g'(x)$

Product Rule:

If y = uv where u and v are functions of x, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In function notation:

If f(x) = g(x)h(x) then:

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

Quotient Rule:

If y = u/v where u and v are functions of x, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

In function notation:

If f(x) = g(x)/h(x) then:

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

Differentiating Exponential and Logarithms

If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

If $y = a^{kx}$, where k is a real constant and a>0, then $\frac{dy}{dx} = a^{kx}k\ln a$

What do I need to be able to do?

By the end of this chapter you should be able to:

- Locate roots of f(x)=0 by considering sign changes
- Use iteration to find an approximation to the equation f(x)=0
- Use the Newton Raphson method to find approximations to the solutions of equations in the form f(x)=0
- Use numerical methods to solve problems in context

Y13 – Chapter 10 Numerical Methods

Key words:

- Root – Where a function equals zero
- Continuous function – The function does not 'jump' from one value to another. If the graph of a function has a vertical asymptote between two points then the function is not continuous in the interval between the two points.

Locating Roots

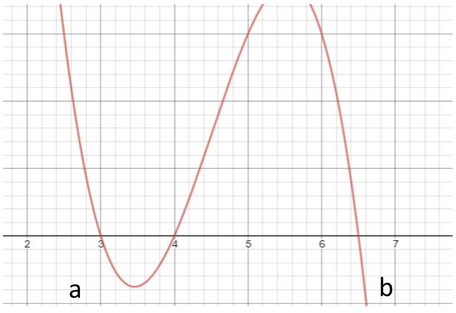
You can sometimes show the existence of a root within a given interval by showing that the function changes sign in that interval

Eg Show that $f(x) = x^3 - 4x^2 + 3x + 1$ has a root between x=1.4 and x=1.5

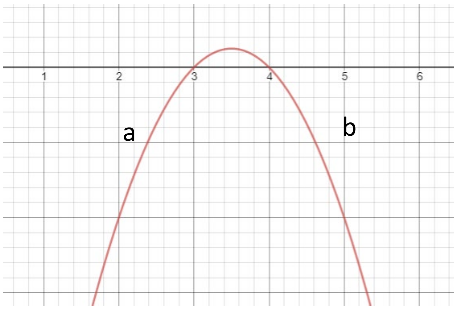
$f(1.4) = (1.4)^3 - 4(1.4)^2 + 3(1.4) + 1 = 0.104$
$f(1.5) = (1.5)^3 - 4(1.5)^2 + 3(1.5) + 1 = -0.125$

The function is continuous and there is a change of sign between 1.4 and 1.5 so there is at least one root in this interval

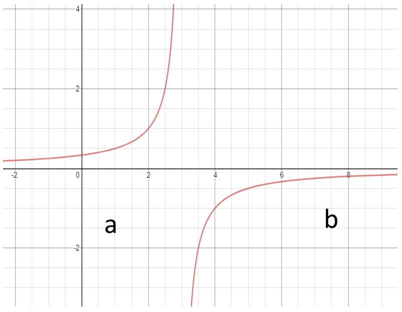
Watch out! The change of sign doesn't mean exactly one root and the absence of a sign change does not necessarily mean that a root does not exist



There are multiple roots within the interval [a, b] – odd number of roots



There are multiple roots within the interval [a, b] but a sign change does not occur – even number of roots



There is a vertical asymptote within the interval [a, b] – a sign change occurs but no roots

Iteration

To solve an equation in the form f(x) = 0 using iteration first rearrange f(x)=0 into the form x=g(x) and then use the iterative formula:

$$x_{n+1} = g(x_n)$$

Some iterations will converge and some will diverge
Successful iterations can be shown on staircase or cobweb diagrams

The Newton-Raphson Method

An alternative way to find roots using differentiation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Y13 – Chapter 11 Integration

What do I need to be able to do?

By the end of this chapter you should be able to:

- Integrate standard mathematical function
- Use trigonometric identities in integration
- Use the reverse chain rule
- Integrate functions by substitution, by parts and using partial fractions
- Find the are under a curve using integration
- Use the trapezium rule
- Solve differential equations and model with differential equations

Key words:

- Differential equation – An equation with a function and one or more of its derivatives

Reverse Chain Rule

Functions in the form f(ax+b):

$$\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$$

To integrate expressions in the form $\int k\frac{f'(x)}{f(x)}dx$, try ln|f(x)| and differentiate to check and adjust the constant as necessary

To integrate functions in the form $\int kf'(x)(f(x))^n\,dx$, try $(f(x))^{n+1}$ and differentiate to check and adjust the constant as necessary

Standard integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c; \quad n \neq -1$$

$$\int e^x dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

You can use trigonometric identities to replace expressions that cannot be integrated with one that can

Eg $\int \tan^2 x\,dx$

$$1 + \tan^2 x \equiv \sec^2 x$$

So

$$\tan^2 x \equiv \sec^2 x - 1$$

So

$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$ which we can integrate using a standard integral

Partial Fractions and Integration by substitution

You can integrate by substitution by choosing a function ($u(x)$) (which can be differentiated) to help you to integrate a tricky function. You need to substitute all $x's$ (including dx) with terms involving u.

You can also use partial fractions in order to integrate algebraic fractions

Differential Equations

When $\frac{dy}{dx} = f(x)g(y)$ you can solve it by separating the variables as follows:

$$\int \frac{1}{g(y)} dy = \int f(x)\,dx$$

When you integrate to solve a differential equation you still need to include a constant of integration – this will give the general solution.

Integration by Parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Area Bounded by Two Curves

$$Area\ of\ R = \int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Limit Notation

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_a^b f(x)\delta x$$

The Trapezium Rule

$$\int_a^b y\,dx \approx \frac{1}{2}h(y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n)$$

Pure Maths Year 2

Y13 – Chapter 12 Vectors

What do I need to be able to do?

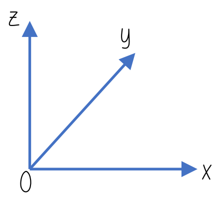
By the end of this chapter you should be able to:

- Understand 3D Cartesian coordinates
- Use vectors in three dimensions
- Use vectors to solve geometric problems
- Model 3D motion in mechanics with vectors

Key words:

- Coplanar vectors – Vectors in the same plane
- Magnitude– The size of the vector

3D Coordinates



When visualising 3D coordinates, think of the x and y axis drawn on a flat surface with the z axis sticking up from the flat surface.

The distance from the origin to the point (x, y, z) is:

$$\sqrt{x^2 + y^2 + z^2}$$

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Parallel Vectors in 3D

If a, b, and c are 3D, non-coplanar vectors (not in the same plane) then you can compare coefficients on both sides of an equation:

Eg

If

$$\boldsymbol{pi} + \boldsymbol{qj} + \boldsymbol{rk} = \boldsymbol{ui} + \boldsymbol{vj} + \boldsymbol{wk}$$

Then: p=u, q=v and r=w

3D Vectors

Unit vectors along the x, y and z axes are denoted by I, j and k respectively

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

For any 3D vector $\boldsymbol{pi} + \boldsymbol{qj} + \boldsymbol{rk} = \begin{pmatrix}p\\q\\r\end{pmatrix}$

Magnitude and Direction

Vector a = $\boldsymbol{pi} + \boldsymbol{qj} + \boldsymbol{rk}$

Magnitude of vector a:

$$|\boldsymbol{a}| = \sqrt{p^2 + q^2 + r^2}$$

Direction of vector a:

The angle with the x-axis: $\cos\theta_x = \frac{p}{|\boldsymbol{a}|}$

The angle with the y-axis: $\cos\theta_y = \frac{q}{|\boldsymbol{a}|}$

The angle with the z-axis: $\cos\theta_z = \frac{r}{|\boldsymbol{a}|}$

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